Problems on General Relativity: 12* optional

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Problem 1. Consider the extremal case of the Kottler spacetime ("Schwarzschild–de Sitter") as the value of ΛM^2 reaches the upper limit, that is

$$g^{\mathrm{K}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{2}$$

$$r \geq 0, \quad \Lambda \geq 0, \quad 9\Lambda M^2 = 1.$$

1. Show, that the function f can be written in the following way

$$f(r) = -\frac{\Lambda}{3r}(r - r_{\mathrm{e}})^2(r + 2r_{\mathrm{e}}) \tag{3}$$

$$r_{\mathrm{e}} := \frac{1}{\sqrt{\Lambda}}. \tag{4}$$

2. Write the metric tensor in a new, ϵ dependent coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$, defined as follows

$$t =: \frac{\tilde{t}}{\epsilon}, \qquad r =: r_{\mathrm{e}} + \epsilon\tilde{r}, \qquad \tilde{\theta} = \theta,\ \tilde{\phi} = \phi, \tag{5}$$

that is

$$g^{\mathrm{K}} = g_{\tilde{t}\tilde{t}}d\tilde{t}^2 + g_{\tilde{r}\tilde{r}}d\tilde{r}^2 + g_{\tilde{\theta}\tilde{\theta}}d\tilde{\theta}^2 + g_{\tilde{\phi}\tilde{\phi}}d\tilde{\phi}^2. \tag{6}$$

3. Show, that there exist all the limits

$$g^{(0)}_{\tilde{x}\tilde{x}} := \lim_{\epsilon\to 0} g_{\tilde{x}\tilde{x}}, \qquad x = \tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi}. \tag{7}$$

4. and that they define a metric tensor (Near Horizon Geometry)

$$g^{\mathrm{NHG}} := g^{(0)}_{\tilde{t}\tilde{t}}d\tilde{t}^2 + g^{(0)}_{\tilde{r}\tilde{r}}d\tilde{r}^2 + g^{(0)}{}_{\tilde{\theta}\tilde{\theta}}d\tilde{\theta}^2 + g^{(0)}{}_{\tilde{\phi}\tilde{\phi}}d\tilde{\phi}^2 = -\Lambda\tilde{r}^2d\tilde{t}^2 + \frac{d\tilde{r}^2}{\Lambda\tilde{r}^2} + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2)) \tag{8}$$

5. Can you argue, that the metric tensor g^{NHG} is not isometric to g^{K}, however its Ricci tensor still satisfies the Einstein equation

$$R^{\mathrm{NHG}}_{\mu\nu} = \Lambda g^{\mathrm{NHG}}_{\mu\nu}. \tag{9}$$

Problem 2. Consider a fluid with energy density ρ, pressure p and 4-velocity u. Its stress-energy tensor is

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(u^\mu u^\nu + g^{\mu\nu}), \tag{10}$$

where g is the spacetime metric tensor. Show, that the equation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{11}$$

amounts to the following system of equations

$$\nabla_\mu\left((\rho + p)\, u^\mu\right) = u^\mu\partial_\mu p. \tag{12}$$

$$(\rho + p)a^\mu = -(u^\mu u^\nu + g^{\mu\nu})\partial_\nu p \tag{13}$$

$$a := \nabla_u u$$

(the equation of continuity and the force acting on a fluid particle, respectively.)
Hint: contract (11) with u_ν and with $(u^\mu u_\nu + g^\mu{}_\nu)$, respectively, and notice, that the latter tensor is the projection onto the space orthogonal to u. Remember, that $u^\mu u_\mu = -1$.

Problem 3. Let $M_{\rm matt}$ be a spacetime endowed with a metric tensor

$$g^{\rm matt} = -d\tau^2 + a^2(\tau)\left(d\chi^2 + s^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{14}$$

where s denotes one of the following three functions

$$s(\chi) = \chi, \quad \text{or} \quad s(\chi) = \sin\chi, \quad \text{or} \quad s(\chi) = \sinh\chi. \tag{15}$$

Consider a surface $S_{\rm matt} \subset M_{\rm matt}$ such that

$$\chi = \chi_0 = \text{const}, \tag{16}$$

parametrised by free values (τ, θ, ϕ).

Let $M_{\rm Sch}$ be a spacetime equipped with the Schwarzschild metric tensor

$$g^{\rm Sch} = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{17}$$

$$\tag{18}$$

Consider a surface $S_{\rm Sch} \subset M_{\rm Sch}$

$$S_{\rm Sch} = \{(t(\tau), r(\tau), \theta, \phi) : \tau, \theta, \phi \;\; \text{are free}\}. \tag{19}$$

1. calculate the metric tensor

   $$q^{\rm matt} = q^{\rm matt}{}_{ij} dx^i dx^j, \qquad (x^1, x^2, x^3) = (\tau, \theta, \phi) \tag{20}$$

   induced in $S_{\rm matt}$ and

   $$q^{\rm Sch} = q^{\rm Sch}{}_{ij} dx^i dx^j, \qquad (x^1, x^2, x^3) = (\tau, \theta, \phi) \tag{21}$$

   induced in $S_{\rm Sch}$.

2. Derive the conditions on $t(\tau), r(\tau), a(\tau)$ and χ_0 upon which

   $$q^{\rm matt}{}_{ij} = q^{\rm Sch}{}_{ij}. \tag{22}$$

3. Show that curves $\tau \mapsto (\tau, \chi_0, \theta_0, \phi_0) \in S_{\rm matt}$ are geodesic in $M_{\rm matt}$.

4. Assume, that curves $\tau \mapsto (t(\tau), r(\tau), \theta_0, \phi_0) \in S_{\rm Sch}$ are geodesic in $M_{\rm Sch}$. Show that the consistency condition is

   $$\left(\frac{da}{d\tau}\right)^2 = \frac{A}{a} + B, \qquad A, B = \text{const}, \qquad \text{A} > 0. \tag{23}$$

5. Think of glueing the region of $M_{\rm matt}$ bounded by $S_{\rm matt}$ into $M_{\rm Sch}$ or vice versa. Consider the cases $\frac{da}{d\tau} > 0$ and $\frac{da}{d\tau} < 0$, respectively.